Chaplygin–like gas and branes in black hole bulks

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Abstract

We explore the possibility to locate a brane in black hole backgrounds. We study explicitly the cases of BHTZ and Schwarzschild–anti de Sitter (AdS) black holes. Our result is that branes cannot be supported by brane tension alone and it is necessary to introduce other forms of matter on the brane. We find classes of perfect fluid solutions obeying to peculiar state equations. For the case of BHTZ bulk geometry the state equation takes exactly the form of a "Chaplygin gas", which is relevant in the brane context. In the Schwarzschild-AdS case we find new state equations which reduce to the Chaplygin form when the brane is located near the horizon.

1 Introduction

In this letter we will study branes in black hole bulks. We will consider mainly two examples, namely BHTZ [1] and anti-de Sitter–Schwarzschild [2] black holes, where we study classes of branes which are naturally adapted to the black hole geometry at hand.

The result we will find is that in both cases there is no possible choice of brane tension so that the geometry under consideration solves the corresponding Einstein equations in the bulk spacetime. It is necessary to go beyond the scheme of Randall and Sundrum [3] by introducing matter on the brane which we will do by making use of perfect fluids. From a geometrical viewpoint, the reason for this new situation is that either the geometry (in the Schwarschild-AdS case) or the foliations (in the BHTZ case) of the bulks we are

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going to study are less symmetric than in the well-studied cases of [3, 4]. This excludes the possibility of writing the metric as a genuine warped manifold [5, 6]. Rather, we are considering a more general setting, that of "multi-warped" manifolds, in which different directions on the brane are multiplied by different functions of the transverse coordinate.

In the Schwarzschild-AdS case there is an exceptional choice of brane location where the possibility exists to consider Einstein equations including only pure brane tension as in [3] and no other type of matter. This particular solution has also appeared recently in [7]. However in our treatment it will appear clearly that this fact is rather a coincidence because the required equality between energy and opposite of the pressure is numerical but not functional.

The type of matter found in our investigation are worth commenting. Both in the BHTZ and Schwarzschild-AdS cases it is possible to identify state equations for the fluids on the branes which are universal (in the geometry considered) in the sense that they are not depending on the choice of brane location. The matter populating branes located in BHTZ bulks has a state equation that is particularly simple: the pressure has to be inversely proportional to minus the (positive) energy density. A fluid satisfying such a state equation is called "Chaplygin gas" . This result is rather interesting since this type of fluid is precisely of interest in the d-brane context [8, 9, 10, 11]. In the AdS-Schwarschild case the state equations we have found are more complicate, but retain the Chaplygin gas form near the horizon.

2 BHTZ black holes

In this section we concentrate on BHTZ black holes. These holes are most easily constructed by taking quotients of the three-dimensional anti–de Sitter spacetime

$$AdS_3 = \{x \in \mathbb{R}^4 : \quad {x^0}^2 - {x^1}^2 - {x^2}^2 + {x^3}^2 = l^2\}, \tag{1}$$

w.r.t. discrete groups of translations along suitable Killing vector fields [1]. For this purpose, a two-parameter class of coordinate systems for AdS_3 is constructed in the following way [1]:

$$x^0 = l\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \cosh\left(\frac{r_+\phi}{l} - \frac{r_-t}{l^2}\right),$$

$$x^1 = l\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \sinh\left(\frac{r_+\phi}{l} - \frac{r_-t}{l^2}\right),$$

$$x^2 = l\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \cosh\left(\frac{r_+ t}{l^2} - \frac{r_- \phi}{l}\right),$$

$$x^3 = l\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \sinh\left(\frac{r_+ t}{l^2} - \frac{r_- \phi}{l}\right), \qquad (2)$$

with $r > r_+ \geq r_-$. In these coordinates the AdS metric is written as follows:

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 - \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 - r^2\left(d\phi - \frac{r_+ r_-}{r^2 l} dt\right)^2. \qquad (3)$$

BHTZ black holes are then obtained by identifying those points parametrized by $\phi + 2\pi k$ with integer k [1].

Let us introduce a new coordinate χ as follows:

$$\cosh\frac{2\chi}{l} = \frac{(r^2 - r_+^2) + (r^2 - r_-^2)}{r_+^2 - r_-^2}, \qquad (4)$$

which gives

$$d\chi = \frac{l r \, dr}{\sqrt{r^2 - r_+^2}\sqrt{r^2 - r_-^2}}. \qquad (5)$$

The use of the coordinate χ recasts the metric in a form which is convenient for a brane "insertion":

$$ds^2 = \frac{B^2 \sinh^2\frac{2\chi}{l}}{A + B\cosh\frac{2\chi}{l}} \frac{dt^2}{l^2} - \left(A + B\cosh\frac{2\chi}{l}\right)\left(\frac{\sqrt{A^2 - B^2}}{A + B\cosh\frac{2\chi}{l}} \frac{dt}{l} - d\phi\right)^2 - d\chi^2, \qquad (6)$$

where

$$2A = r_+^2 + r_-^2, \qquad 2B = r_+^2 - r_-^2. \qquad (7)$$

It is clear from this expression that the coordinate χ can play the role of a transverse space-like coordinate in the bulk as in [3]. The difference w.r.t the situation studied in [3] is that the metric is now multi-warped, i.e. the remaining terms of ds^2 are multiplied by different functions of χ.

Let us consider now a brane located at $\chi = \chi_b$. The geometry we are going to study is made up by two slices of the BHTZ black hole geometry glued together at the brane with a symmetry (orbifold-type) condition analogous to that exploited in [3]. This geometry is

described by the following components of the metric tensor (w.r.t. the chosen coordinate system):

$$g_{tt} = Bl^{-2}\cosh\frac{2u}{l} - Al^{-2},$$

$$g_{\phi\phi} = -B\cosh\frac{2u}{l} - A,$$

$$g_{t\phi} = l^{-1}\sqrt{A^2 - B^2},$$

$$g_{\chi\chi} = -1. \tag{8}$$

where we have introduced $u = \chi_b + |\chi - \chi_b|$. The contravariant components are the following:

$$g^{tt} = \frac{A + B\cosh\frac{2u}{l}}{B^2\sinh^2\frac{2u}{l}}\,l^2,$$

$$g^{\phi\phi} = \frac{A - B\cosh\frac{2u}{l}}{B^2\sinh^2\frac{2u}{l}},$$

$$g^{t\phi} = \frac{\sqrt{A^2 - B^2}}{B^2\sinh^2\frac{2u}{l}},$$

$$g^{\chi\chi} = -1. \tag{9}$$

The metric induced on the brane is obtained as the restriction of the components given in Eq. (8) (first fundamental form):

$$h_{tt} = Bl^{-2}\cosh\frac{2\chi_b}{l} - Al^{-2},$$

$$h_{\phi\phi} = -B\cosh\frac{2\chi_b}{l} - A,$$

$$h_{t\phi} = l^{-1}\sqrt{A^2 - B^2}. \tag{10}$$

Since the components of the bulk metric are not continuosly differentiable, Chistoffel's symbols have finite jumps at $\chi = \chi_b$. These jumps give rise to the following Einstein tensor on the brane:

$$G_{ij}^{(b)} = \frac{4}{l^2}\coth\frac{2\chi_b}{l}\,h_{ij} + \tilde{G}_{ij}^{(b)}, \tag{11}$$

where

$$\tilde{G}_{tt}^{(b)} = -\frac{2B}{l^3\sinh\frac{2\chi_b}{l}}, \qquad \tilde{G}_{\phi\phi}^{(b)} = \frac{2B}{l\sinh\frac{2\chi_b}{l}}, \qquad \tilde{G}_{t\phi}^{(b)} = 0. \tag{12}$$

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Outside the brane the Einstein tensor obviously coincides with the anti–de Sitter one. We have therefore the following structure:

$$G_{\mu\nu} = G_{\mu\nu}^{AdS} + \delta_\mu^i \delta_\nu^j G_{ij}^{(b)} \delta(\chi - \chi_b). \qquad (13)$$

Let us first of all consider the non-rotating case. This corresponds to $2A = 2B = r_+^2$ and the metric is static: $g_{t\phi} = 0$ and therefore $h_{t\phi} = 0$. However, due to the presence of the term $\widetilde{G}^{(b)}$, the brane tension alone is not enough to solve the corresponding Einstein equations and it is necessary to introduce matter on the brane. We will concentrate on perfect fluids (but other types of matter are worth investigating). The following relations on the brane are thus obtained:

$$G_{tt}^{(b)} = (\varepsilon + p)u_t^2 - p \, h_{tt},$$
$$G_{\phi\phi}^{(b)} = (\varepsilon + p)u_\phi^2 - p \, h_{\phi\phi},$$
$$G_{t\phi}^{(b)} = (\varepsilon + p)u_t u_\phi = 0. \qquad (14)$$

The third equation calls for a static fluid, corresponding to the condition $u_\phi = 0$. The first two equations in the previous array become simply $G_{tt}^{(b)} = \varepsilon \, h_{tt}$ and $G_{\phi\phi}^{(b)} = -p \, h_{\phi\phi}$, which are solved by

$$\varepsilon = +\frac{2}{l}\tanh\frac{\chi_b}{l}$$
$$p = -\frac{2}{l}\coth\frac{\chi_b}{l} \qquad (15)$$

It is possible to write an equation of state which includes all the previous solutions:

$$p\varepsilon = -\frac{4}{l^2}. \qquad (16)$$

This state equation is universal in the sense that it is the only one which has the same form for any brane location. It corresponds formally to the so-called Chaplygin gas which has recently raised a certain interest [8, 9]. The pressure of this fluid is negative and inversely proportional to the energy density. The interesting fact is that this type of fluid can be obtained also from the Nambu-Goto action for d-branes moving in a $(d+2)$-dimensional spacetime in the light-cone parametrization [10, 11]. For a string embedded in a three-dimensional flat spacetime the argument is so simple that we may reproduce it: the relevant Hamiltonian is

$$H = \frac{1}{2}\int \left[P^2 + (\partial_\sigma x)^2\right] d\sigma,$$

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where x is the only transversal coordinate. Let us introduce euristically the density [11] $\varepsilon(x) = (\partial_\sigma x)^{-1}$ and the velocity $v = P$. One of the equations of motion is then current conservation:

$$\partial_t \varepsilon + \partial_x(\varepsilon v) = 0.$$

The other gives

$$\dot{P} = \partial_\sigma^2 x = \partial_\sigma x \partial_x(1/\varepsilon)$$

that is

$$\varepsilon \dot{P} = \varepsilon(\partial_t + v\partial_x)v = \partial_x(1/\varepsilon)$$

which is Euler equation for a fluid such that $p = -1/\varepsilon$.

Let us pass now to a short discussion of the rotating case. Einstein equations now give the following relations on the brane:

$$\begin{aligned}
G_{tt}^{(b)} &= (\varepsilon + p)u_t^2 - p\, h_{tt}, \\
G_{\phi\phi}^{(b)} &= (\varepsilon + p)u_\phi^2 - p\, h_{\phi\phi}, \\
G_{t\phi}^{(b)} &= (\varepsilon + p)u_t u_\phi - p\, h_{t\phi}.
\end{aligned} \tag{17}$$

This system should be supplemented by the normalization condition

$$h^{tt}u_t^2 + 2h^{t\phi}u_t u_\phi + h^{\phi\phi}u_\phi^2 = 1. \tag{18}$$

After some algebraic computation the following solution is obtained:

$$u_t^2 = \frac{r_+^2}{l^2}\sinh^2\frac{\chi_b}{l}, \quad u_\phi^2 = r_-^2 \sinh^2\frac{\chi_b}{l}. \tag{19}$$

Now we can proceed as before and solve for the fluid energy density and pressure. The solution, and consequently the equation of state, are exactly the same as in the static case, see Eqs. (15,16). The corresponding full energy-momentum tensor is now not diagonal.

The coincidence of the energy and pressure obtained in the rotating and nonrotating cases deserves a further explanation. A closer look to Eq. (4) reveals that the brane does not really depends on the choice of r_+ and r_- and consequently of the corresponding Killing vectors. Indeed, from the viewpoint of the space in which the anti-de Sitter universe is embedded, the brane can be decribed by the following equations:

$$\begin{cases}
x^{0^2} - x^{1^2} + x^{2^2} - x^{3^2} = l^2 \cosh\frac{2\chi_b}{l} \\
x^{0^2} - x^{1^2} - x^{2^2} + x^{3^2} = l^2
\end{cases} \tag{20}$$

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This means that we could have studied such a brane in the anti-de Sitter bulk *before* making identifications along the Killing vectors. The simplest choice amounts to studying a bulk spacetime with interval

$$ds^2 = \sinh^2 u \ dt^2 - \cosh^2 u \ d\phi^2 - d\chi^2 \tag{21}$$

which describes the geometry of two regions of the anti de Sitter manifold glued together at a two-dimensional brane having the topology of a plane. The BHTZ procedure then changes both the bulk and the brane topology but has no consequence on the energy-momentum tensor of the matter living on the brane.

It is interesting to note that the brane is obtained in Eq. (20) as the intersection of two AdS spacetimes embedded in $\mathbb{R}^4$. Of course this is peculiar to two-dimensional branes. The present study could in principle be extended to other dimensionalities [12], but in this case the fluid would not be isotropic.

3 Schwarzschild-AdS black holes

We now examine the situation for the Schwarzschild-AdS metric [2], as recently studied also in [7]. We do this in the usual case in which the bulk spacetime is five-dimensional and the brane four-dimensional. The Schwarzschild-AdS metric [2] is given by

$$ds^2 = \left(1 + \frac{r^2}{l^2} - \frac{2M}{r^2}\right) dt^2 - \frac{1}{1 + \frac{r^2}{l^2} - \frac{2M}{r^2}} \ dr^2 - r^2 d^2\Omega^{(3)}, \tag{22}$$

where $d^2\Omega^{(3)}$ is the metric of a unit three-dimensional spherical surface. As before we may introduce a radial coordinate χ by

$$\cosh \frac{2\chi}{l} = \frac{2r^2 + l^2}{\sqrt{8Ml^2 + l^4}}, \tag{23}$$

and insert a "brane" at $\chi = \chi_b(r_b)$ by requiring the same symmetry properties of the metric w.r.t. χ_b as in the previous section. The brane is now substantially a copy of a Einstein static universe. The Einstein tensor includes a term proportional to $\delta(\chi - \chi_b)$, which should be matched by matter on the brane so as to compensate it. Again, a perfect fluid with the following energy density and pressure does the job:

$$\varepsilon = \frac{6}{r_b}\sqrt{1 + \frac{r_b^2}{l^2} - \frac{2M}{r_b^2}}, \tag{24}$$

$$p = -\frac{4\left(1 + \frac{3r_b^2}{2l^2} - \frac{M}{r_b^2}\right)}{r_b\sqrt{1 + \frac{r_b^2}{l^2} - \frac{2M}{r_b^2}}}. \tag{25}$$

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The state equation now depends on the black hole mass M:

$$p = -\frac{\varepsilon}{3} - \frac{24}{\varepsilon l^2} - \frac{12}{\varepsilon r_b^2(\varepsilon)} \tag{26}$$

where $r_b^2(\varepsilon)$ is obtained by solving Eq. (24). Two regimes are possible: if $0 < \varepsilon < 6/l$ there is only a solution, given by

$$r_b^2(\varepsilon) = -\frac{18l^2}{36 - \varepsilon^2 l^2} + \sqrt{\left(\frac{18l^2}{36 - \varepsilon^2 l^2}\right)^2 + \frac{72Ml^2}{36 - \varepsilon^2 l^2}} \tag{27}$$

When $\varepsilon > 6/l$ the following two solutions are possible:

$$r_b^2(\varepsilon) = \frac{18l^2}{\varepsilon^2 l^2 - 36} \pm \sqrt{\left(\frac{18l^2}{\varepsilon^2 l^2 - 36}\right)^2 - \frac{72Ml^2}{\varepsilon^2 l^2 - 36}} \tag{28}$$

In this regime there is a special value $r_b^2 = 4M$ where to locate the brane. Here, the energy density ε has a maximum value

$$\varepsilon_{\max} = \frac{6}{l}\sqrt{1 + \frac{l^2}{8M}}, \tag{29}$$

and the two solutions coalesce; beyond this value of the energy density there is no solution (see figure 1).

Since in this very case one also has that $\varepsilon = -p$ numerically, the fluid can be alternatively interpreted as tension of the brane as in the original work by Randall and Sundrum [3]. This particular situation has been recently considered by [7]. However, in our scheme it appears clearly that this case is rather special and in any other location one is obliged to add other matter contributions on the brane.

It is worthwhile to remark that the state equation (26) reduces to the Chaplygin equation when the brane is located near the horizon. For $r_b^2 > 4M$ there exists also the possibility to think of two distinguished types of matter on the brane, namely brane tension and a standard fluid having both energy density and pressure positive.

When l tends to infinity we are describing a brane in a Schwarzschild bulk. Eqs. (24), (25), (26) and (28) can be easily reduced to this case; the situation is substantially similar to that we have described above in the Schwarschild-AdS case and we do not reproduce the results.

On the other side, when M is set to zero the metric (22) reduces to the metric of an empty AdS spacetime and we are back to the Randall-Sundrum setting. However,
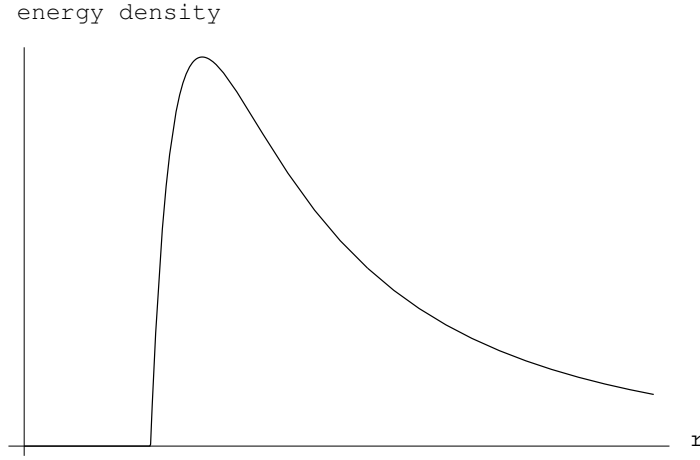
Figure 1: Energy density ε as a function of the brane location for finite positive values of l and M. The energy density ε vanishes when the brane is located at the horizon; ε has a maximum at $r_b^2 = 4M$.

by putting a brane at some value $r = r_b$ of the chosen coordinates we are exploring the different case in which the brane has the four-geometry of an Einstein static universe. The constant r_b has therefore also the meaning of the radius of this universe. Also in this case we need to add matter beyond brane tension and this because of the brane geometry. Energy and pressure of the matter populating this universe have a comparatively simple form:

$$\varepsilon = \frac{6}{r_b}\sqrt{1 + \frac{r_b^2}{l^2}}, \tag{30}$$

$$p = -\frac{4\left(1 + \frac{3r_b^2}{2l^2}\right)}{r_b\sqrt{1 + \frac{r_b^2}{l^2}}}. \tag{31}$$

The state equation is now as follows:

$$p = -\frac{2\varepsilon}{3} - \frac{12}{\varepsilon l^2}. \tag{32}$$

When the radius r_b of the brane tends to infinity we are finally back to the original Randall-Sundrum case [3]. Indeed in this limit we have that numerically

$$\varepsilon = \frac{6}{l} = -p. \tag{33}$$

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Eq. (33) recovers the relation betweeen the brane tension and the bulk cosmological term given in [3] (up to notations).

Finally, the state equation(32) can be easily generalized to the $(n + 2)$-dimensional anti de Sitter bulk spacetime as follows:

$$p = -\frac{\varepsilon(n-1)}{n} - \frac{4n}{\varepsilon l^2}. \tag{34}$$

4 Concluding remarks

We have shown that the study of branes in black hole bulks requires the use of multi-warped manifolds and there is the necessity of matter on the branes to set up acceptable Einstein equations.

The matter we find is rather interesting. It is a "Chaplygin gas" in BHTZ situation and a more complicate type of gas in the Schwarschild-AdS case, which however retains the "Chaplygin" form near the horizon. In both cases the matter we find is of "cosmological" type: a (positive) cosmological can be thought as a fluid having (constant) positive energy density and negative pressure. This type of matter may also be of interest in the context of "quintessential" expansion models and, from an observative viewpoint, it may have a connection with the by now accepted existence of a positive acceleration of the universe expansion [13, 14].

It would be interesting to explore if there exist an interpretation for the equation of state we have found in the Schwarschild-AdS case from the viewpoint of a theory of extended objects.

Acknowledgements

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